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Crossover behaviour and effective critical exponents in isotropic and anisotropic Heisenberg systems

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Abstract. Renormalization group methods are used to determine, to second order in $\epsilon = 4-d$, the scaling function describing the crossover from Gaussian to Heisenberg behaviour in the susceptibility of an isotropic n-component spin system. The results are used in conjunction with an earlier Feynman graph calculation to obtain an $O(\epsilon^2)$ representation of the n-to-m-component susceptibility crossover function, and the corresponding effective exponents, for an anisotropic n-component system.

1. Introduction

The application of renormalization group (RG) techniques to the study of crossover phenomena has recently received considerable attention. Developing the techniques pioneered by Wilson (1971) and Riedel and Wegner (1974), Nelson and Rudnick (1975) have shown how Wilson's RG recursion relations (Wilson and Kogut 1974) may be exploited to yield expressions for crossover scaling functions. These are the functions which describe the way in which the critical behaviour of a system, with a small 'universality-class-changing' perturbation, evolves from 'primary' character (typical of the unperturbed system) far from T_c to its 'secondary' character (which reflects the perturbation) in the asymptotic regime. These techniques have been used to analyse the behaviour of n-component vector models with a dominant m-component spin anisotropy (Nelson and Domany 1976, Kosterlitz 1976) and the behaviour of a Heisenberg system with dipole–dipole interactions (Bruce *et al* 1976).

All such calculations have, so far, been restricted to lowest order in $\epsilon \equiv 4-d$, largely as a result of problems associated with the additional irrelevant variables that enter the Wilson RG equations at $O(\epsilon^2)$. As in the calculation of critical exponents and amplitude ratios, these problems may be by-passed, to some extent, witth the aid of Wilson's (1972) Feynman graph (FG) methods, which may be utilized to yield the ϵ expansions of crossover scaling functions directly. The $O(\epsilon^2)$ expansion of the susceptibility crossover function for the aforementioned anisotropic spin system was calculated in this way by one of us (Bruce 1975, to be referred to as I). Unfortunately, this approach suffers from the same defect as the FG calculation of the equation of state of the n-vector model (Brézin *et al* 1973), namely that it yields a representation of the appropriate scaling function that consists of a series of logarithmically divergent terms (equation (3.19) of I) whose singularities (associated with the secondary critical behaviour—or the behaviour on the co-existence curve in the equation-of-state problem) have to be interpreted as the ϵ expansion of some power law behaviour which can be reconstructed only with the aid of additional information.

The purpose of the present work is twofold. Firstly, as described in § 2, we use Zinn-Justin's (1973) formulation of the RG to yield a representation of the susceptibility scaling function for the Gaussian–Heisenberg crossover problem to *second* order in ϵ. This approach is characterized by RG equations (bearing some similarity to those of a limiting case of Wilson's equations (Bruce *et al* 1974)) which have the merit that the equations for the troublesome higher-order transients are decoupled.

The Gaussian–Heisenberg crossover problem is itself of some physical relevance in the contexts of tricritical phenomena in isotropic n-vector systems, and the polymer ($n=0$ component) problem (Burch and Moore 1976). In addition, however, as we describe in the second main portion of this work, contained in § 3, the results of § 2 may be utilized to obtain, in conjunction with the results of I, a complete description of the n-to-m-component crossover behaviour in the susceptibility of the anisotropic spin system. Specifically, our calculations reveal how the aforementioned logarithms, obtained in the ϵ expansion of the scaling function found in I, should be exponentiated to yield a satisfactory representation of the limiting behaviour of the crossover function. The results of these calculations, like those of § 2, are summarized with the aid of the 'effective-exponent' concept, introduced by Riedel and Wegner (1974).

2. The Gaussian–Heisenberg crossover

2.1. Renormalization group equations

The problem we consider in this section can be described by the Ginzburg–Landau–Wilson effective Hamiltonian

$$\mathcal{H}=\int \mathrm{d}^d x\left(\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}(m_{0c}^2+t)\phi^2+\tfrac{1}{2}\Lambda^{-2}(\nabla^2\phi)^2+\frac{u\Lambda^\epsilon}{4!}(\phi^2)^2\right). \qquad (2.1)$$

Here ϕ is an n-component field and the O(n) invariance of $\mathcal{H}$ corresponds to an isotropic Heisenberg system. The bare mass term has been split up into m_{0c}^2+t where m_{0c}^2 is the bare mass of the critical theory and t is a measure of the reduced temperature $(T-T_c)$. The cut off Λ is implemented by the fourth-order derivatives in (2.1) and the bare coupling is written in terms of a dimensionless coupling u and a dimensional factor Λ^ϵ, since Λ^{-1} is the microscopic length scale controlling dimensional parameters in $\mathcal{H}$. As usual, $\epsilon=4-d$ where d is the dimension of space.

When $u=0$, equation (2.1) is the Gaussian Hamiltonian of a free-field theory, and its critical exponents are those of mean-field theory. For any positive u (we do not consider $u<0$ in this paper), (2.1) describes a Heisenberg system, with Heisenberg exponents differing from mean-field exponents if $d<4$, no matter how small u is. The dimensionless quantity $u\Lambda^\epsilon t^{-\epsilon/2}$ controls how rapidly the behaviour crosses over from Gaussian ('primary') to Heisenberg ('secondary') character, as $t/\Lambda^2\to 0$, for small u.

The behaviour of thermodynamic functions as $t/\Lambda^2\to 0$ can be obtained using RG and ϵ-expansion methods. The particular form of RG we shall use is that described by Zinn-Justin (1973). Consider the proper vertex functions $\Gamma^{(N)}(q;\, u,\, t,\, \Lambda)$. These are the sums of all one-particle irreducible graphs with N external legs; q represents the N external momenta and N spin indices have been suppressed. These functions obey the

RG equation

$$\left(\Lambda\frac{\partial}{\partial\Lambda}+\beta(u)\frac{\partial}{\partial u}-(\gamma_4(u)-\gamma_3(u))t\frac{\partial}{\partial t}-\tfrac{1}{2}N\gamma_3(u)\right)\Gamma^{(N)}(q;u,t,\Lambda)=\Delta\Gamma^{(N)} \tag{2.2}$$

where $\Delta\Gamma^{(N)}$ is smaller than $\Gamma^{(N)}$ by terms of order (t/Λ^2), (q^2/Λ^2) up to powers of $\ln(t/\Lambda^2)$, $\ln(q/\Lambda)$ in an ϵ expansion. If we can neglect such terms in the critical region $t \ll \Lambda^2$, $q \ll \Lambda$, then we obtain a homogeneous equation

$$\left(\Lambda\frac{\partial}{\partial\Lambda}+\beta(u)\frac{\partial}{\partial u}-(\gamma_4(u)-\gamma_3(u))t\frac{\partial}{\partial t}-\tfrac{1}{2}N\gamma_3(u)\right)\Gamma^{(N)}(q;u,t,\Lambda)=0 \tag{2.3}$$

obeyed by vertex functions in which only the leading terms (in the above sense) have been retained.

The same approximation on vertex functions is made in Wilson's (1972) FG method, where, however, the $\Gamma^{(N)}$ are expressed in terms of the exact inverse susceptibility r instead of the reduced temperature t. Thus instead of calculating the $\Gamma^{(N)}$ perturbatively with a bare propagator $(q^2+t+\Lambda^{-2}q^4)^{-1}$, and a mass counterterm m_{0c}^2 as suggested by equation (2.1), it is convenient to use the tabulations for Wilson's method given by Nickel (1972, 1974), exploiting the self-consistency condition for the self-energy

$$t=\Sigma(0,r)-\Sigma(0,0) \tag{2.4}$$

to eliminate r in favour of t. Straightforward calculations then yield the results

$$\begin{aligned}\Gamma^{(2)}(0;u,t,\Lambda)&\equiv r\\ &=t\{1+\tfrac{1}{6}(n+2)u[\ln(t/\Lambda^2)+1-\tfrac{1}{4}\epsilon\ln^2(t/\Lambda^2)]\\ &\quad+\tfrac{1}{36}(n+2)u^2[(n+5)\ln^2(t/\Lambda^2)+3(n+2)\ln(t/\Lambda^2)+2(n+2)\\ &\quad-3(4+\pi^2-8\lambda)]\}+O((u,\epsilon)^3)\end{aligned} \tag{2.5}$$

$$\Gamma^{(2)}(q;u,0,\Lambda)=q^2\{1-\tfrac{1}{18}(n+2)u^2[\ln(q/\Lambda)+\tfrac{1}{4}-\tfrac{2}{3}\lambda]\}+O((u,\epsilon)^3) \tag{2.6}$$

and

$$\begin{aligned}\Gamma^{(4)}&(0;u,t,\Lambda)\\ &=\Lambda^{\epsilon}u(1+\tfrac{1}{6}(n+8)u\{[1+\tfrac{1}{6}u(n+2)]\ln(t/\Lambda^2)\\ &\quad+\tfrac{1}{6}u(n+2)+2-\epsilon[\tfrac{1}{4}\ln^2(t/\Lambda^2)+\tfrac{1}{2}\ln(t/\Lambda^2)]\}\\ &\quad+\tfrac{1}{36}(n^2+6n+20)u^2[\ln^2(t/\Lambda^2)+4\ln(t/\Lambda^2)+4]\\ &\quad+\tfrac{1}{9}(5n+22)u^2[\tfrac{1}{2}\ln^2(t/\Lambda^2)+\ln(t/\Lambda^2)-2-\tfrac{1}{2}\pi^2+4\lambda])+O((u,\epsilon)^4).\end{aligned} \tag{2.7}$$

All graphs with up to two loops are included in these expressions, including constants at the highest order in u (depending on π and $\lambda \simeq 1{\cdot}17$) which will contribute only to non-universal scale factors at this order in ϵ. For simplicity, a factor $K_d \equiv \pi^{d/2}\Gamma(\tfrac{1}{2}d)^{-1}(2\pi)^{-d}$ from each loop integral is absorbed into u.

The fact that these three vertex functions obey equation (2.3) yields three simultaneous equations which are readily solved perturbatively for the functions β, γ_4 and γ_3.

All dependence on $\ln(t/\Lambda^2)$ disappears and one finds

$$\beta(u) = -\epsilon u + \tfrac{1}{3}(n+8)u^2(1+\tfrac{1}{2}\epsilon) - \tfrac{1}{3}(3n+14)u^3 + O((u, \epsilon)^4) \tag{2.8}$$

$$\gamma_4(u) = -\tfrac{1}{3}(n+2)u(1+\tfrac{1}{2}\epsilon) + \tfrac{1}{3}(n+2)u^2 + O((u, \epsilon)^3) \tag{2.9}$$

$$\gamma_3(u) = \tfrac{1}{18}(n+2)u^2 + O((u, \epsilon)^3). \tag{2.10}$$

As in renormalized perturbation theory (see Brézin *et al* 1976 for a review) we utilize these results in conjunction with (2.3), which we integrate by the method of characteristics. Thus we define $\tau \equiv \ln \Lambda$ and introduce functions $u(\tau)$ and $t(\tau)$ by

$$\frac{du(\tau)}{d\tau} = \beta(u(\tau)) \tag{2.11}$$

$$\frac{dt(\tau)}{d\tau} = -t(\tau)(\gamma_4(u(\tau)) - \gamma_3(u(\tau))) \tag{2.12}$$

with initial condition $u(0) = u$, $t(0) = t$ corresponding to the 'physical' system with coupling u, reduced temperature t and cut-off $\Lambda = e^0 = 1$. Then (2.3) has the formal solution

$$\Gamma^{(N)}(q; u, t, 1) = \exp\left(-\tfrac{1}{2}N \int_0^\tau \gamma_3(u(\tau'))\, d\tau'\right)\Gamma^{(N)}(q; u(\tau), t(\tau), e^\tau). \tag{2.13}$$

We shall exploit this equation in the following section, where the critical behaviour, $t \ll 1$, of the left-hand side is obtained by chóosing a value of τ such that the vertex function on the right-hand side is that of a system far from criticality.

2.2. *The susceptibility crossover scaling function*

We now proceed to implement the results of the preceding subsection—to calculate the susceptibility, given by

$$\chi^{-1} \equiv r = \Gamma^{(2)}(0; u, t, 1) \tag{2.14}$$

as a function of u and t, to second order in ϵ. The first step in this calculation is the integration of equations (2.11) and (2.12) for $u(\tau)$ and $t(\tau)$, using the functions β, γ_4 and γ_3 as defined in equations (2.8)–(2.10), and with the initial conditions $u(0) = u$, $t(0) = t$. The solution for $u(\tau)$ may be written in the form

$$\left(\frac{u^* - u(\tau)}{u^* - u}\right)^{-\epsilon/\lambda_H} \left(\frac{u(\tau)}{u}\right)^{-\epsilon/\lambda_G} = e^{\epsilon\tau} \tag{2.15}$$

where u^* is the solution of $\beta(u^*) = 0$,

$$u^* = \frac{3\epsilon}{n+8}\left(1 - \frac{(n^2 - 2n - 20)}{2(n+8)^2}\epsilon\right) + O(\epsilon^3) \tag{2.16}$$

while λ_H and λ_G are defined by

$$\lambda_G = \epsilon$$

$$\lambda_H = -\epsilon + \frac{(9n+42)}{(n+8)^2}\epsilon^2 + O(\epsilon^3). \tag{2.18}$$

We remark that (2.15) is a solution of (2.8) and (2.11) within the approximations inherent in the former equation; that is, $du/d\tau$ as obtained from (2.15) is in accord with (2.8) and (2.11) to within terms $O((\epsilon, u)^4)$. In the same spirit, equation (2.12) may be integrated to yield

$$t(\tau)=t\left(\frac{u^*-u(\tau)}{u^*-u}\right)^{\rho}\left(1+\frac{(n+2)(13n+44)}{6(n+8)^2}(u(\tau)-u)\right) \tag{2.19}$$

where

$$\rho=\frac{(n+2)}{(n+8)}\left(1+\frac{(31n+128)}{2(n+8)^2}\epsilon\right). \tag{2.20}$$

We note, for future reference, the equality

$$\rho=[(1/\nu)-2]/\lambda_H+O(\epsilon^2) \tag{2.21}$$

where λ_H is given by (2.18) and ν is the usual Heisenberg exponent (Wilson 1972).

The second step requires the evaluation of the right-hand side of equation (2.13) to the appropriate order in ϵ. Evidently $\Gamma^{(2)}(0; u(\tau), t(\tau), e^{\tau})$ is given directly by equation (2.5) with the replacements $u\to u(\tau)$, $t\to t(\tau)$, $\Lambda\to e^{\tau}$ while the exponential prefactor has already been obtained in the calculation leading to (2.19). Then, substituting for $t(\tau)$ using (2.19), one obtains

$$\begin{aligned}\chi^{-1}&=\Gamma^{(2)}(0; u, t, 1)\\ &=t\left(\frac{u^*-u(\tau)}{u^*-u}\right)^{\rho'}\left(1+\frac{2(n+2)(n+3)}{(n+8)^2}(u(\tau)-u)\right)\\ &\quad\times\{1+\tfrac{1}{6}u(\tau)(n+2)[\ln(t(\tau)\,e^{-2\tau})+1-\tfrac{1}{4}\epsilon\ln^2(t(\tau)\,e^{-2\tau})]\\ &\quad+\tfrac{1}{36}u^2(\tau)(n+2)[(n+5)\ln^2(t(\tau)\,e^{-2\tau})+3(n+2)\ln(t(\tau)\,e^{-2\tau})+2(n+2)\\ &\quad-3(4+\pi^2-8\lambda)]\}\end{aligned} \tag{2.22}$$

where

$$\rho'=\frac{(n+2)}{(n+8)}\left(1+\frac{15(n+4)}{(n+8)^2}\epsilon\right)+O(\epsilon^2). \tag{2.23}$$

Again, we note that ρ' may be expressed as

$$\rho'=(1-\gamma)/(\lambda_H\nu) \tag{2.24}$$

where γ is the Heisenberg susceptibility exponent with the expansion, which we quote for completeness (Wilson 1972),

$$\gamma=1+\frac{(n+2)}{2(n+8)}\epsilon+\frac{(n+2)(n^2+22n+52)}{4(n+8)^3}\epsilon^2+O(\epsilon^3). \tag{2.25}$$

We remark that, although we have retained terms of order $u^2(\tau)$ and $\epsilon u(\tau)$ in the curly brackets in (2.22), such terms will contribute (at this order) only to non-universal critical amplitudes and could in principle have been neglected.

Finally, we need to employ a matching condition such that the perturbation expansion in $u(\tau)$ in equation (2.22) can be trusted. A suitable choice is

$$t(\tau)=e^{2\tau}. \tag{2.26}$$

This condition realises the conventional renormalization group aim of relating the physics of a critical system to that of a new system which is non-critical, and whose behaviour can (as we have supposed) be calculated in perturbation theory (Wilson and Kogut 1974, Riedel and Wegner 1974). Using (2.15), (2.19) and (2.22) and introducing

$$p \equiv u(\tau)/u^* \tag{2.27}$$

one obtains a representation of the susceptibility in parametric form:

$$\chi^{-1} = Z(u)t\left(\frac{1-p}{1-(u/u^*)}\right)^{\rho'}\left(1+\frac{(n+2)(n^2+28n+100)}{6(n+8)^2}u^*p\right)+O(\epsilon^3) \tag{2.28}$$

with

$$(1-p)^{(\epsilon/\lambda_H)+(\epsilon\rho/2)}p = t^{-\epsilon/2}(u/u^*)[1-(u/u^*)]^{(\epsilon/\lambda_H)+(\epsilon\rho/2)}. \tag{2.29}$$

We have separated out an analytic, non-universal prefactor $Z(u)$ defined by

$$Z(u) = 1 - u\frac{2(n+2)(n+3)}{(n+8)^2} + u^2\left(\frac{(n+2)}{36}(2n-8-3\pi^2+24\lambda) + \frac{(n+2)^2(n+3)(n^2+28n+100)}{3(n+8)^4}\right). \tag{2.30}$$

Equations (2.28) and (2.29) may be cast into a more compact form with the introduction of non-linear scaling fields $\mathcal{T}$ and U, defined by

$$\mathcal{T} = tZ(u)[1-(u/u^*)]^{-\rho'} \tag{2.31}$$

$$U = (u/u^*)(Z(u))^{\epsilon/2}[1-(u/u^*)]^{\gamma\epsilon/(2\lambda_H\nu)}. \tag{2.32}$$

Making use of these substitutions, and recalling (2.21) and (2.24), one finds

$$\chi^{-1} = \mathcal{T}(1-p)^{\rho'}\left(1+\frac{(n+2)(n^2+28n+100)}{6(n+8)^2}u^*p\right) \tag{2.33}$$

with

$$(1-p)^{\epsilon/(2\lambda_H\nu)}p = U\mathcal{T}^{-\epsilon/2}. \tag{2.34}$$

These equations together give a parametric representation of the susceptibility crossover scaling function

$$\chi = \mathcal{T}^{-1}\Psi(U/\mathcal{T}^{\epsilon/2}) \tag{2.35}$$

with $\mathcal{T}$ and U playing the roles of non-linear scaling fields associated with the Gaussian fixed point. (See, however, the discussion below.) We remark that this representation is 'correct to $O(\epsilon^2)$' in the sense that, when re-expanded in ϵ, it is in accord with a direct perturbation expansion to this order. We also note that at $O(\epsilon)$ these results are in accord with the calculations of Nelson and Rudnick (1975).

As anticipated, Ψ has a small-z power series expansion, which can be determined solving (2.33) for p as a power series in $z \equiv U\mathcal{T}^{-\epsilon/2}$. The singularity in the crossover scaling function occurs at $z=\infty$, and the behaviour of the susceptibility in this asymptotic (large-z) region may be obtained by solving (2.34) for $1-p$ as a power series in $z^{2\lambda_H\nu/\epsilon}$. Inspection of (2.33) shows that the divergence of the susceptibility is indeed governed by γ (equation (2.25)), and that the next to leading terms are characterized, as anticipated, by the exponent

$$\bar{\omega} \equiv -\lambda_H \nu = \tfrac{1}{2}\epsilon - \frac{(68+8n-n^2)}{4(n+8)^2}\epsilon^2 + O(\epsilon^3). \tag{2.36}$$

For the purposes of the calculation described in the following section, it is also useful to have a closed-form representation for the susceptibility with the correct analytic structure in this asymptotic regime. This may be obtained by rewriting (2.28) as

$$\chi^{-1} = Z(u)t^{\gamma}(pu^*/u)^{2(\gamma-1)/\epsilon}\left(1+\frac{(n+2)(n^2+28n+100)}{6(n+8)^2}u^*p\right) \tag{2.37}$$

and writing, as a solution for (2.29),

$$p = (1-t^{\bar{\omega}}[1-(u^*/u)]\{(u/u^*)+t^{\bar{\omega}}[1-(u/u^*)]\}^{[1-(2\bar{\omega}/\epsilon)]})^{-1}. \tag{2.38}$$

While formally correct within an ϵ expansion this equation has the correct analytic structure only for the asymptotic (Heisenberg) regime; for a solution that has the correct analytic structure in both Gaussian and Heisenberg regimes one must use the parametric form, equations (2.33) and (2.34).

Two remarks are now in order. The reader may note that the renormalization group equation (2.3) is valid only for the leading singular terms in $\Gamma^{(N)}$, and yet one extends the solution (2.13) of (2.3) into a region $t(\tau) = e^{2\tau}$, in which the non-leading terms in the RHS of (2.13) are as large as the leading terms we retain! However, this is not erroneous, because equation (2.3) applies formally to the leading singular terms in $\Gamma^{(N)}$, regardless of whether the 'non-leading' singular terms are as large as the 'leading' singular terms. The only approximation lies in the assumption that the full *critical* vertex function $\Gamma^{(2)}(0; u, t, 1)$ $(t \ll 1)$ can be approximated by its leading singular terms, neglecting the fast transients, which are softer by powers of t of order 1. (The dubious reader can circumvent this problem by using renormalized perturbation theory; an explicit check shows that, at $O(\epsilon^2)$, and with the renormalization conditions of Brézin *et al* (1976), the equations are exactly the same as those derived above with Λ replaced by the renormalization point μ, and u and t replaced by renormalized quantities.)

Lastly, we turn to consider the status of the 'non-linear scaling field' $\mathcal{T}$ introduced in (2.31). According to this equation, $\mathcal{T}$ is proportional to the (true) reduced temperature $t \equiv (T-T_c(u))/T_c(u)$. It is known, however, that the transition temperature, $T_c(u)$ is a *non-analytic* function of the coupling constant u at fixed non-zero ϵ (Symanzik 1973, Sugar and White 1974) even though it is an *analytic* function of ϵ at fixed u (Balian 1975). Specifically, $T_c(u)$ (and hence t itself) contains terms of the form $u^{2/\epsilon}$. In consequence, $\mathcal{T}$ does not have the analytic properties required of a scaling field of the Gaussian fixed point, but should have the form (for small u)

$$\mathcal{T} = \mathcal{T}_0[1-c(u\mathcal{T}_0^{-\epsilon/2})^{2/\epsilon}] \tag{2.39}$$

where $\mathcal{T}_0$ *is* analytic in u. The susceptibility (2.23) is thus expressible as a function of $u\mathcal{T}_0^{-\epsilon/2}$. *This* function, however, will *not* have the anticipated analytic form for small $z_0 \equiv u\mathcal{T}_0^{-\epsilon/2}$, but will contain terms of the form $z_0^{2/\epsilon}$. This suggests, therefore, that the calculation leading to (2.32) may have failed to pick up some compensating non-analytic terms. It is not clear how serious this defect is; however, since these terms cannot make an appearance within the (strict) framework of an ϵ expansion upon which this work is based, equation (2.33) is, at the very least, a consistent representation of the susceptibility to second order in ϵ.

2.3. *The effective susceptibility exponent*

In figure 1 we display our results for the effective exponent (Riedel and Wegner 1974)

$$\gamma_{\rm eff} = -\frac{\partial \ln \chi}{\partial \ln \mathscr{T}} \tag{2.40}$$

obtained from the parametric form (2.33) and (2.34) for the Ising model ($n = 1$) in three dimensions ($\epsilon = 1$). The curves are drawn with U set equal to unity and $\mathscr{T}$ normalized such that $\Psi(z) = 1 - z + O(z^2)$. The exponents in (2.32) and (2.33) are chosen so that $1/\bar{\omega}$ and γ are correct to $O(\epsilon^2)$. (Thus $(\gamma - 1)/\bar{\omega}$ does contain some extraneous higher powers of ϵ.)

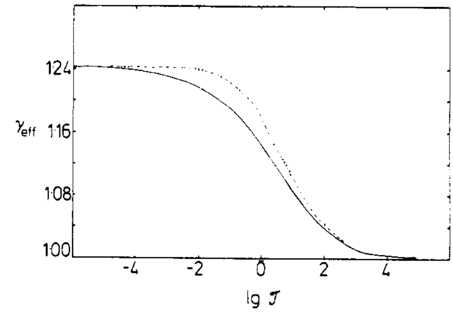

Figure 1. The effective exponent $\gamma_{\rm eff}$ for the Gaussian–Ising crossover. The full curve is the result of the full $O(\epsilon^2)$ calculation. The broken curve represents the results obtained from the phenomenological equations (2.41) and (2.42).

For comparison, we also show the result of a phenomenological calculation (along the lines suggested by Riedel and Wegner (1974)) based on the equations (cf (2.11) and (2.12))

$$\frac{du(\tau)}{d\tau} = -u(\tau) + u^2(\tau) \tag{2.41}$$

$$\frac{dt(\tau)}{d\tau} = Au(\tau)t(\tau) \tag{2.42}$$

where A is chosen to give the exponent γ correct to second order in ϵ; χ is then identified with $t(\tau)$ in the limit $\tau \to -\infty$. There is a surprising difference from the exact $O(\epsilon^2)$ results, principally because the correction to the scaling exponent $\bar{\omega}$, which governs how rapidly $\gamma_{\rm eff}$ assumes its asymptotic value, is 0·5 in the phenomenological approach and closer to 0·3 in the full calculation. This difference may be detectable in the ($n = 0$) polymer problem—see Burch and Moore (1976).

3. Crossover behaviour in the anisotropic Heisenberg system

3.1. *Preliminaries*

As indicated in the introduction, the solution of the Gaussian–Heisenberg crossover problem, described in the preceding section, may be exploited to yield, in conjunction with the results obtained in I, a complete description of the susceptibility crossover in the anisotropic Heisenberg system. It is this calculation that we now describe, beginning with a summary of the results of I and the philosophy of the analysis described in the following subsection.

Briefly (for full details the reader is referred to I) the system we consider is an n-component Heisenberg model (of the form (2.1)) with an additional anisotropic exchange term

$$\mathcal{H}_g = -\tfrac{1}{2}g\int d^dx\left(\frac{1}{m}\sum_{\alpha=1}^{m}\phi_\alpha^2 - \frac{1}{n-m}\sum_{\alpha=m+1}^{n}\phi_\alpha^2\right) \tag{3.1}$$

where g is a measure of the anisotropy and is assumed small. The perturbation (3.1) results (for $g>0$) in an asymptotic m-component critical behaviour, the crossover to this behaviour being described (in the case of the susceptibility) by the scaling form (Pfeuty *et al* 1974)

$$\chi^{-1} = t^{\gamma(n)}(1-y)^{\gamma(m)}P^{-1}(y) \tag{3.2}$$

where $\gamma(n)$ and $\gamma(m)$ are the n-component and m-component Heisenberg exponents (defined by (2.25)), t is the reduced temperature

$$t \equiv T - T_c(g=0) \tag{3.3}$$

and y is defined as $\bar{x}/\dot{x}$, with $\bar{x} \propto g/t^{\phi}$. Here ϕ is the appropriate crossover exponent (Wilson 1972) and $\dot{x}$ locates the singularity of the crossover scaling function; its ϵ expansion was obtained in I (equation (3.20)). The function $P^{-1}(y)$ is obtained (after some manipulation) from equation (3.19) of I in the form

$$\begin{aligned}
P^{-1}(y) = 1 + \frac{1}{1-y}\bigg\{&\frac{\epsilon}{2(n+8)}\left[(n-m+my)\ln\left(1+\frac{my}{n-m}\right)\right.\\
&\left.-ny\ln\left(\frac{n}{n-m}\right)\right]\left(1+\frac{(n^2+26n+84)}{2(n+8)^2}\epsilon\right)+\frac{(n-m)}{8(n+8)^2}\epsilon^2\\
&\times\left[(n-2m)\left(1+\frac{my}{n-m}\right)\ln^2\left(1+\frac{my}{n-m}\right)\right.\\
&\left.+\frac{2nmy}{n-m}\ln\left(1+\frac{my}{n-m}\right)\ln\left(\frac{n}{n-m}\right)\right]\\
&+\frac{n(m+2)(n-m)}{4(n+8)^2(m+8)}\epsilon^2\ln(1-y)\left[y\ln\left(\frac{n}{n-m}\right)-\ln\left(1+\frac{my}{n-m}\right)\right]\\
&-\frac{n^2y}{8(n+8)^2}\epsilon^2\ln^2\left(\frac{n}{n-m}\right)\bigg\}+\ln(1-y)\left[\frac{(m+2)(m-n)}{2(n+8)(m+8)}\epsilon\right.\\
&+\frac{(n+2)(n^2+22n+52)-(n-m)(n^2+26n+84)}{4(n+8)^3}\epsilon^2\\
&-\frac{(m+2)(m^2+22m+52)}{4(m+8)^3}\epsilon^2\\
&\left.+\frac{m(m+5)(n-m)}{2(n+8)^2(m+8)}\epsilon^2\ln\left(1+\frac{my}{n-m}\right)\right]
\end{aligned}$$

$$+\ln^2(1-y)\left(\frac{(2m^2-mn+4m+4)}{8(n+8)^2}\epsilon^2+\frac{(m+2)^2}{8(m+8)^2}\epsilon^2-\frac{(m+2)^2}{4(m+8)(n+8)}\epsilon^2\right)$$

$$-\frac{(n-m)}{2(n+8)^2}\left[I_1\left(\frac{1-y}{4\{1+[my/(n-m)]\}}\right)-I_1(\tfrac{1}{4})\right]+O(\epsilon^3) \qquad (3.4)$$

where the function I_1 was defined in equation (3.7) of I.

The appearance of the $\ln(1-y)$ terms in (3.4) is a manifestation of the fact that (as discussed in I) the crossover scaling function has built into it corrections to the asymptotic (m-component) scaling behaviour. This is true (in an experimental situation) even if t and g are small enough that corrections to the primary scaling behaviour are vanishingly small, or (in the theoretical analogue) when these corrections are specifically eliminated by the choice of coupling constant in the perturbation expansion that leads to (3.4). Of the secondary correction to scaling terms, it is only those 'slow transients' characterized by exponents $O(\epsilon)$ that lead to the truly divergent logarithms in (3.4). In certain crossover problems (the cubic–Heisenberg crossover, discussed by Aharony (1973), is one example) there may be more than one such transient. In this problem, however, it is clear from RG recursion-relation insights into the evolution of the perturbation causing the crossover, that there should be only the one—namely that associated with the leading correction to scaling behaviour for an isotropic m-component system. It is this observation that we exploit to determine the limiting behaviour of (3.4). Specifically we shall make use of the fact that, in the $y\to 1$ limit, the expression $(1-y)^{\gamma(m)}P^{-1}(y)$ (with the prefactor also ϵ-expanded), must match on to a direct ϵ expansion for the (inverse) susceptibility of an isotropic m-component system—i.e. (2.5), but with n replaced by m, and u and t replaced by an effective coupling constant and an effective temperature variable

$$n\to m \qquad (3.5a)$$

$$u\to u(y) \qquad (3.5b)$$

$$t\to f(y)(1-y) \qquad (3.5c)$$

where we demand that $u(y)$ and $f(y)$ are analytic functions of y, with well defined limits as $y\to 1$. An exponentiated form for $P^{-1}(y)$ may then be read off from (2.37) and (2.38) with the appropriate substitutions.

It must be remarked that, were we to adhere strictly to the philosophy expounded above, we would demand only that this matching should be possible in the asymptotic $y\to 1$ limit, thence determining two constants $u(1)$ and $f(1)$. The use of the effectively temperature-dependent quantities $u(y)$ and $f(y)$, however, enables us to incorporate those terms in the function $P^{-1}(y)$ (equation (3.4)), which vanish as $(1-y)\ln(1-y)$. It is possible that such terms are manifestations of fast transients, and should, in fact, be exponentiated to give correction to scaling terms vanishing as $(1-y)^{1+O(\epsilon)}$. The result we shall obtain should, however, offer a perfectly adequate representation of the susceptibility throughout the crossover regime, having the merits that (a) it will certainly give a rigorously correct account of the asymptotic behaviour (unlike the unexponentiated form (3.4)) and (b) it has an expansion in ϵ that is correct to $O(\epsilon^2)$.

3.2. Calculations and results

It is simply a matter of rather lengthy algebra to implement the matching procedure outlined above. To $O(\epsilon^2)$ we find that the effective coupling constant $u(y)$ may be written as

$$u(y)=u^*(n)+\frac{3(n-m)}{(n+8)^2}\epsilon^2\left[1+\tfrac{1}{2}\ln\left(1+\frac{my}{n-m}\right)\right.$$
$$\left.-\frac{1}{m+2}\sum_{p=1}^{\infty}\left(\frac{1-y}{4\{1+[my/(n-m)]\}}\right)^p\frac{\Gamma(p)\Gamma(\frac{3}{2})}{\Gamma(p+\frac{3}{2})}\right] \tag{3.6}$$

where $u^*(n)$ is the n-component fixed point value (2.16). The final term has its origins in the non-elementary integral I_1 in equation (3.4), which may be written as

$$I_1\left(\frac{1-y}{4\{1+[my/(n-m)]\}}\right)=\tilde{I}_1\left(\frac{1-y}{4\{1+[my/(n-m)]\}}\right)+\text{terms analytic near } y=1 \tag{3.7}$$

with the non-analytic terms defined by

$$\tilde{I}_1\left(\frac{1-y}{4\{1+[my/(n-m)]\}}\right)$$
$$=\ln(1-y)\left[\tfrac{1}{2}\ln(1-y)-\ln\{1+[my/(n-m)]\}-2\right.$$
$$\left.+\sum_{p=1}^{\infty}\left(\frac{1-y}{4\{1+[my/(n-m)]\}}\right)^p\frac{\Gamma(p)\Gamma(\frac{3}{2})}{\Gamma(p+\frac{3}{2})}\right]. \tag{3.8}$$

The normalization of the temperature scale may be written conveniently as

$$f(y)=\tilde{f}(y)[1+\tfrac{1}{6}u(y)(m+2)+\tfrac{1}{36}u^2(y)(m+2)(2m-8-3\pi^2+24\lambda)]^{-1} \tag{3.9}$$

where $\tilde{f}(y)$ is defined by

$$\tilde{f}(y)=1+\frac{(m+2)^2}{4(n+8)^2}\epsilon^2+\frac{1}{1-y}\left\{-y\left[\frac{n\epsilon}{2(n+8)}\ln\left(\frac{n}{n-m}\right)\right.\right.$$
$$+\frac{n}{4(n+8)^2}\epsilon^2(n^2+24n+68-mn-8m)\ln\left(\frac{n}{n-m}\right)$$
$$\left.+\frac{n^2}{8(n+8)^2}\epsilon^2\ln^2\left(\frac{n}{n-m}\right)\right]$$
$$+\frac{(n-m)}{2(n+8)}\epsilon\left(1+\frac{my}{n-m}\right)\ln\left(1+\frac{my}{n-m}\right)\left(1+\frac{n^2+24n+68-mn-8m}{2(n+8)^2}\epsilon\right)$$
$$-\frac{(n-m)}{2(n+8)^2}\epsilon^2(1-y)\left[I_1\left(\frac{1-y}{4\{1+[my/(n-m)]\}}\right)\right.$$
$$\left.-\tilde{I}_1\left(\frac{1-y}{4\{1+[my/(n-m)]\}}\right)-I_1(\tfrac{1}{4})\right]$$
$$+\frac{(n-m)}{8(n+8)^2}\epsilon^2\left[(n-2m)\left(1+\frac{my}{n-m}\right)\ln^2\left(1+\frac{my}{n-m}\right)\right.$$
$$\left.\left.+\frac{2nmy}{n-m}\ln\left(1+\frac{my}{n-m}\right)\ln\left(\frac{n}{n-m}\right)\right]\right\}. \tag{3.10}$$

Then, using (3.6), (3.9) and (3.10) and making the substitutions (3.5), one obtains from (2.37) the result

$$P^{-1}(y) = (Q(y))^{2(\gamma(m)-1)/\epsilon}\left(1+\frac{(m+2)(m^2+28m+100)}{2(m+8)^2(n+8)}\epsilon(Q(y)-1)\right)h(y) \tag{3.11}$$

where $h(y)$ is defined by

$$h(y) = \tilde{f}(y)\left\{1-\frac{(m+2)^2}{4(n+8)^2}\epsilon^2+\frac{(m+2)\epsilon^2}{4(n+8)(m+8)(1-y)}\right.$$
$$\left.\times\left[(n-m+my)\ln\left(1+\frac{my}{n-m}\right)-ny\ln\left(\frac{n}{n-m}\right)\right]\right\}. \tag{3.12}$$

The variable $Q(y)$ is defined by an equation analogous to (2.38), namely

$$Q(y)(u(y)/u^*(m))$$
$$=\left[1-[\tilde{f}(y)(1-y)]^{\bar{\omega}(m)}[1-(u^*(m)/u(y))]\right.$$
$$\left.\times\left(\frac{u(y)}{u^*(m)}+[\tilde{f}(y)(1-y)]^{\bar{\omega}(m)}[1-(u(y)/u^*(m))]\right)^{1-2\bar{\omega}(m)/\epsilon}\right]^{-1}. \tag{3.13}$$

Equation (3.11), together with (3.6), (3.10), (3.12) and (3.13) defines our exponentiated representation of the function (3.4). It is a reassuring, though lengthy, task to check that the ϵ expansion of (3.11) does indeed coincide with (3.4). One may also check that these results are in accord with the $O(\epsilon)$ calculations of Nelson and Domany (1976) and Kosterlitz (1976).

Finally, we note that the universal amplitude $\dot{X}$ defined by (Pfeuty *et al* 1974)

$$\dot{X} = \lim_{y\to 1} P(y) \tag{3.14}$$

is given by (3.11) as

$$\dot{X} = \left(\frac{m+8}{n+8}\right)^{(m+2)/(m+8)}\left[1-\frac{3(n-m)\epsilon}{(n+8)(m+8)}\ln\left(\frac{n}{n-m}\right)\right.$$
$$-\frac{(m+2)(m^2+22m+52)}{2(m+8)^3}\epsilon\ln\left(\frac{n+8}{m+8}\right)-\frac{(m+2)(n^2-2n-20)}{2(n+8)^2(m+8)}\epsilon$$
$$-\frac{(m+2)(30mn+120n-m^3-36m^2-84m+160)}{2(m+8)^3(n+8)}\epsilon$$
$$\left.-\frac{m^2-2mn-4n-4m}{2(m+8)(n+8)}\epsilon\right]. \tag{3.15}$$

It may be remarked that this result is compatible with the spherical model (n, $m\to\infty$ limit) result, which one may readily derive as

$$\dot{X} = \left(\frac{m}{n}\frac{2}{2-\epsilon}\right)^{2/(2-\epsilon)} \tag{3.16}$$

This differs from a result of Oppermann (1975); the discrepancy may be partially attributed to this author's use of a scaling variable that is effectively $y^{1/\phi}$, rather than y.

3.3. *Discussion*

The most sensitive test of the results presented in the preceding section is afforded by comparison of the representation (equations (3.11) onwards) for $P(y)$ with the series expansion calculations of Pfeuty *et al* (1974) and Singh and Jasnow (1975a, b). Figure 2 shows this function calculated for the case $n = 2$, $m = 1$ and compared with two of the series expansion approximants examined by Singh and Jasnow (1975a). Evidently there is good agreement between the ϵ expansion result and the second of these approximants. This is in accord with the conclusions of I, where it was noted that the $O(\epsilon^2)$ values of the amplitude $\dot{x}$ are in most satisfactory agreement with those series results based upon a small argument expansion of the scaling function; indeed, these were the results that were used as a basis for the second series approximant (Singh and Jasnow 1975a). This trend is maintained in the values of the amplitude $\dot{X}$; table 1 shows the values assigned to this amplitude on the basis of the series work compared with those obtained from the result (3.15) and those obtained by evaluating $P(1)$ explicitly from (3.11). The latter (which are in significantly better agreement with the series

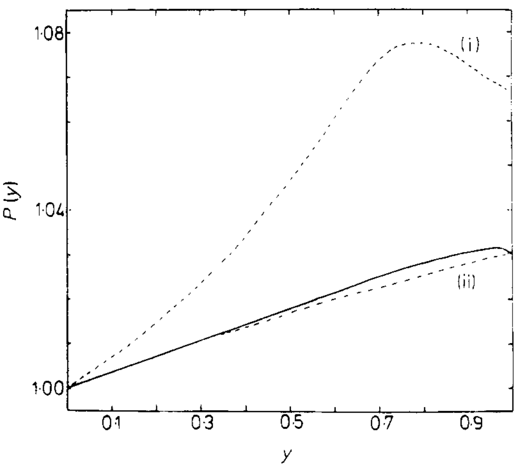

Figure 2. The function $P(y)$ for the $n = 2$ to $m = 1$ component crossover. The full curve is the result of the $O(\epsilon^2)$ expression derived here; the broken curves are the results of two of the approximants obtained in the series work of Singh and Jasnow (1975a).

Table 1. Values of the universal amplitude $\dot{X}$ for n-component spin systems with dominant m-component anisotropy. The ϵ-expansion results are obtained from the evaluation of $P(1)$ using (3.11) and (in parentheses) from the strict $O(\epsilon)$ expression (3.15). The series results are obtained from the work of Pfeuty *et al* (1974) and Singh and Jasnow (1975a, b), using the values obtained from a small $\tilde{x}$ expansion of the scaling function and (in parentheses) from an analysis at finite g, extrapolating to $g = 0$.

n	m	$\dot{X}(n, m)$	
		ϵ expansion	Series expansion
2	1	1·029 (0·973)	1·030 (1·066)
3	1	0·984 (0·919)	1·00 (1·09)
3	2	1·100 (0·998)	1·075 (1·158)

results) differ from the former in that they contain terms of second (and higher) order in ϵ which we cannot strictly retain in the formal expansion (3.15).

From an experimental point of view, potentially the most physically interesting quantity yielded by our analysis is the effective critical exponent γ_{eff}, defined by (2.40) with $\mathcal{T} \equiv t - t_c$ measuring the deviation from the true transition temperature. With the aid of (3.2) it is easy to show that one may write this effective exponent in the form

$$\gamma_{\text{eff}}(n, m) = (1 - y^{1/\phi}) \left(\gamma(n) + \frac{\phi y \gamma(m)}{1-y} + \frac{\phi y P'(y)}{P(y)} \right). \tag{3.17}$$

The resulting function $\gamma_{\text{eff}}(2, 1)$ (with $P(y)$ given by (3.11)) is compared, in figure 3, with the series results of Singh and Jasnow (1975a). We have introduced a reduced temperature variable $\bar{t}$, proportional to $t - t_c$, and defined by

$$\bar{t} \equiv y^{-1/\phi} - 1. \tag{3.18}$$

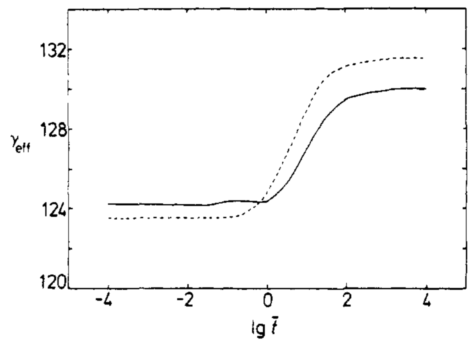

Figure 3. The effective exponent γ_{eff} for the $n = 2$ to $m = 1$ component crossover. The full curve is the result of the $O(\epsilon^2)$ calculation; the broken curve is a result of the series work (Singh and Jasnow 1975a). Note that only the *forms* of these curves may be meaningfully compared; their relative positions on the logarithmic temperature scale are subject to uncertainties involving non-universal scale factors.

The good agreement is to be expected in view of the close correspondence (figure 2) of our result for $P(y)$ with the second series approximant, which was used to generate the series results shown in figure 3.

It is to be remarked that, while the essential form of $\gamma_{\text{eff}}(n, m)$ is determined by the first two terms in the parentheses in (3.17), the third term (evidently the hardest to calculate!) can contribute significant refinements. As a measure of its importance we show in figure 4 the results of two computations of $\gamma_{\text{eff}}(3, 2)$, one ignoring, and the other including this term. Clearly the marked dip present in the former result (and present, to some extent, in the $O(\epsilon)$ calculations of Nelson and Domany (1975)) is suppressed in the correct $O(\epsilon^2)$ calculation, and the effective exponent evolves relatively smoothly between its two limits. This is also true of the Heisenberg–Ising crossover which we therefore do not show.

Finally, we remark that while at present the experimental situation offers little with which to compare the results of our calculations, such a comparison may ultimately be afforded with further investigations of 'spin flop' systems (Rohrer 1975). These systems are describable (see Nelson *et al* 1974, Kosterlitz *et al* 1976) by the anisotropic spin Hamiltonian we have investigated, with the anisotropy parameter g tunable, through adjustment of an applied magnetic field.

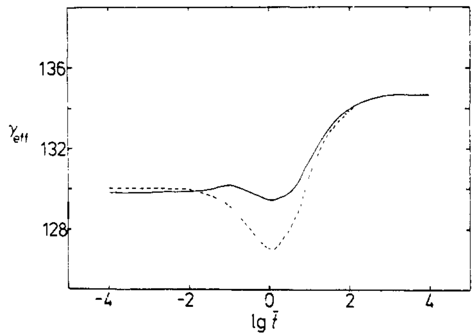

Figure 4. The effective exponent γ_{eff} for the $n = 3$ to $m = 2$ component crossover. The full curve is the result of the full $O(\epsilon^2)$ calculation; the broken curve is the result obtained when the third term in the parentheses in (3.17) is ignored.

4. Concluding remarks

In summary we have obtained a representation of the susceptibility crossover scaling function in anisotropic spin systems, to $O(\epsilon^2)$, using an approach which offers a useful and labour saving synthesis of renormalization group and Feynman graph methods.

In principle it should, of course, be possible to obtain these results entirely within the framework of the RG, avoiding the matching to the FG expansion. However, of the pure RG methods used to obtain scaling functions to $O(\epsilon)$, only that of Horner (1976) would appear to be tractable in an $O(\epsilon^2)$ calculation.

Our method is, of course, applicable to other thermodynamic functions, and to other crossover problems. We shall report elsewhere the results of analogous calculations for the Heisenberg–dipolar crossover, where $O(\epsilon)$ calculations (Bruce *et al* 1976) suggested the existence of an experimentally intriguing dip in the effective susceptibility exponent in the crossover regime. At $O(\epsilon)$ this dip is much more pronounced than its anisotropic spin counterparts, mainly by virtue of the large asymptotic value of γ_{eff}, and of the crossover exponent appropriate for this problem. In view of the experimental situation (review by Bruce *et al* 1976) it is of considerable interest to determine whether this effect is enhanced or diminished by the second-order calculations.

Acknowledgments

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Note added in proof. After completion of this work we received a preprint from I D Lawrie (1976 *J. Phys. A*: *Math. Gen.* **9** 961–73) which uses renormalized perturbation theory to study the Gaussian–Heisenberg crossover problem to order ϵ^2.

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